ON FUZZY $k$-IDEALS IN SEMIRINGS

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Abstract. In this paper, with the notion of fuzzy $k$-ideals of semirings, we discuss and review several results described in [4].

1. Introduction

L. A. Zadeh ([10]) introduced the notion of a fuzzy subset $\mu$ of a set $X$ as a function from $X$ into the closed unit interval $[0, 1]$. The concept of fuzzy subgroups was introduced by A. Rosenfeld ([8]). W. J. Liu ([7]) studied fuzzy ideals in rings. T. K. Dutta and B. K. Biswas ([2, 3]) studied fuzzy ideals, fuzzy prime ideals of semirings, and they defined fuzzy $k$-ideals and fuzzy prime $k$-ideals of semirings and characterized fuzzy prime $k$-ideals of semirings of non-negative integers and determined all its prime $k$-ideals. Recently, Y. B. Jun, J. Neggers and H. S. Kim ([4]) extended the concept of an $L$-fuzzy (characteristic) left (resp., right) ideal of a ring to a semiring $R$, and showed that each level left (resp., right) ideal of an $L$-fuzzy left (resp., right) ideal $\mu$ of $R$ is characteristic iff $\mu$ is $L$-fuzzy characteristic. The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (e.g., [1, 5, 6, 9, 11]). The notion of $k$-ideals and $Q$-ideals ([1]) were applied to construct quotient semirings. In this paper, with the notion of fuzzy $k$-ideals of semirings, we discuss and review several results described in [4].

2. Preliminaries

An algebra $(R; +, \cdot)$ is said to be a semiring ([11]) if $(R; +)$ and $(R; \cdot)$...
are semigroups satisfying $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$. A semiring $R$ may have an identity 1, defined by $1 \cdot a = a = a \cdot 1$, and a zero 0, defined by $0 + a = a = a + 0$ and $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in R$.

From now on we write $R$ and $S$ for semirings. A non-empty subset $I$ of $R$ is said to be a left (resp., right) ideal if $x, y \in I$ and $r \in R$ imply that $x + y \in I$ and $r x \in I$ (resp., $x r \in I$). If $I$ is both left and right ideal of $R$, we say $I$ is a two-sided ideal, or simply, ideal of $R$. A left ideal $I$ of a semiring $R$ is said to be a left $k$-ideal if $a \in I$ and $x \in R$ and if $a + x \in I$ or $x + a \in I$ then $x \in I$. Right $k$-ideal is defined dually, and two-sided $k$-ideal or simply a $k$-ideal is both a left and a right $k$-ideal.

A mapping $f : R \rightarrow S$ is said to be a homomorphism if $f(x + y) = f(x) + f(y)$ and $f(x y) = f(x)f(y)$ for all $x, y \in R$. We note that if $f : R \rightarrow S$ is an onto homomorphism and $I$ is a left (resp., right) ideal of $R$, then $f(I)$ is a left (resp., right) ideal of $S$.

**Definition 2.1** ([2]). A fuzzy subset $\mu$ of a semiring $R$ is said to be a fuzzy left (resp., right) ideal of $R$ if $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(x y) \geq \mu(y)$ (resp., $\mu(x y) \geq \mu(x)$) for all $x, y \in R$. $\mu$ is a fuzzy ideal of $R$ if it is both a fuzzy left and a fuzzy right ideal of $R$.

**Definition 2.2** ([3]). A fuzzy ideal $\mu$ of a semiring $R$ is said to be a fuzzy $k$-ideal of $R$ if

$$\mu(x) \geq \min\{\max\{\mu(x + y), \mu(y + x)\}, \mu(y)\}$$

for all $x, y \in R$. If $R$ is additively commutative then the condition reduces to $\mu(x) \geq \min\{\mu(x + y), \mu(y)\}$ for all $x, y \in R$.

Note that every fuzzy ideal of a ring is a fuzzy $k$-ideal.

**Example 2.3** ([3]). Let $\mu$ be a fuzzy subset of the semiring $N$ of natural numbers defined by

$$\mu(x) := \begin{cases} 0.3, & \text{if } x \text{ is odd}, \\ 0.5, & \text{if } x \text{ is non-zero even}, \\ 1, & \text{if } x = 0. \end{cases}$$

Then $\mu$ is a fuzzy $k$-ideal of $N$. 
Example 2.4 ([3]). Let \( \mu \) be a fuzzy subset of the semiring \( N \) of natural numbers defined by

\[
\mu(x) := \begin{cases} 
1, & \text{if } 7 \leq x, \\
0.5, & \text{if } 5 \leq x < 7, \\
0, & \text{if } 0 \leq x < 5.
\end{cases}
\]

Then it is easy to show that \( \mu \) is a fuzzy ideal of \( N \), but not a fuzzy \( k \)-ideal of \( N \).

Proposition 2.5 ([3]). Let \( I \) be a non-empty subset of a semiring \( R \) and \( \lambda_I \) the characteristic function of \( I \). Then \( I \) is a \( k \)-ideal of \( R \) if and only if \( \lambda_I \) is a fuzzy \( k \)-ideal of \( R \).

3. Main Results

Y. B. Jun, J. Neggers and H. S. Kim ([4]) studied \( L \)-fuzzy ideals in semirings, and T. K. Dutta and B. K. Biswas ([3]) defined the notion of fuzzy \( k \)-ideals in semirings. With the notion of fuzzy \( k \)-ideals of semirings we discuss and review several results described in [4].

Proposition 3.1. A fuzzy subset \( \mu \) of \( R \) is a fuzzy left (resp., right) \( k \)-ideal of \( R \) if and only if, for any \( t \in [0,1] \) such that \( \mu_t \neq \emptyset \), \( \mu_t \) is a left (resp., right) \( k \)-ideal of \( R \), where \( \mu_t = \{ x \in R | \mu(x) \geq t \} \), which is called a level subset of \( \mu \).

Proof. It was proved that a fuzzy subset \( \mu \) is a fuzzy left (resp., right) ideal of \( R \) if and only if for any \( t \in [0,1] \) such that \( \mu_t \neq \emptyset \), \( \mu_t \) is a left (resp., right) ideal of \( R \) (see [4]). Assume that \( \mu \) is a fuzzy \( k \)-ideal of \( R \). Suppose that \( a \in \mu_t \) and \( x \in R \), and \( a + x \in \mu_t \) or \( x + a \in \mu_t \). Then \( \mu(a) \geq t, \mu(a + x) \geq t \) or \( \mu(x + a) \geq t \), and hence \( \max\{\mu(a + x), \mu(x + a)\} \geq t \). Since \( \mu \) is a fuzzy \( k \)-ideal of \( R \), \( \mu(x) \geq \min\{\max\{\mu(a + x), \mu(x + a)\}, \mu(a)\} \), i.e., \( x \in \mu_t \). Hence \( \mu_t \) is a \( k \)-ideal of \( R \).

Conversely, assume \( \mu_t \) is a \( k \)-ideal of \( R \), for any \( t \in [0,1] \) with \( \mu_t \neq \emptyset \). For any \( x, a \in R \), let \( \mu(a) = t_1, \mu(x + a) = t_2, \mu(a + x) = t_3 \) \((t_i \in [0,1]) \). If we let \( t := \min\{\max\{t_2, t_3\}, t_1\} \), then \( a \in \mu_t \) and \( a + x \in \mu_t \) or \( x + a \in \mu_t \). Since \( \mu_t \) is a \( k \)-ideal of \( R \), we have \( x \in \mu_t \), i.e.,
\[\mu(x) \geq \min\{\max\{\mu(x + a), \mu(a + x)\}, \mu(a)\},\] proving that \(\mu\) is a fuzzy \(k\)-ideal of \(R\).

Note that if \(\mu\) is a fuzzy left (resp., right) \(k\)-ideal of \(R\) then the set \(R_{\mu} := \{x \in R|\mu(x) \geq \mu(0)\}\) is a left (resp., right) \(k\)-ideal of \(R\).

**Theorem 3.2.** Let \(I\) be any left (resp., right) \(k\)-ideal of \(R\). Then there exists a fuzzy left (resp., right) \(k\)-ideal \(\mu\) of \(R\) such that \(\mu_t = I\) for some \(t \in [0, 1]\).

**Proof.** If we define a fuzzy subset of \(R\) by

\[\mu(x) := \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}\]

for some \(t \in [0, 1]\), then it follows that \(\mu_t = I\). For a given \(s \in [0, 1]\) we have

\[\mu_s = \begin{cases} \mu_0 (= R) & \text{if } s = 0, \\ \mu_t (= I) & \text{if } s \leq t, \\ \emptyset & \text{if } t < s \leq 1. \end{cases}\]

Since \(I\) and \(R\) itself are left (resp., right) \(k\)-ideals of \(R\), it follows that every non-empty level subset \(\mu_s\) of \(\mu\) is a left (resp., right) \(k\)-ideal of \(R\). By Proposition 3.1, \(\mu\) is a fuzzy left (resp., right) \(k\)-ideal of \(R\), proving the theorem. \(\Box\)

Let \(\mu\) and \(\nu\) be fuzzy subsets of \(R\). We denote that \(\mu \subseteq \nu\) if and only if \(\mu(x) \leq \nu(x)\) for all \(x \in X\), and \(\mu \subset \nu\) if and only if \(\mu \subseteq \nu\) and \(\mu \neq \nu\).

**Theorem 3.3.** Let \(\mu\) be a fuzzy left (resp., right) \(k\)-ideal of \(R\). Then two level left (resp., right) \(k\)-ideals \(\mu_s, \mu_t\) (with \(s < t\) in \([0, 1]\)) of \(\mu\) are equal if and only if there is no \(x \in R\) such that \(s \leq \mu(x) < t\).

**Proof.** Suppose \(s < t\) in \([0, 1]\) and \(\mu_s = \mu_t\). If there exists an \(x \in R\) such that \(s \leq \mu(x) < t\), then \(\mu_t\) is a proper subset of \(\mu_s\), a contradiction. Conversely, suppose that there is no \(x \in R\) such that \(s \leq \mu(x) < t\). Note that \(s < t\) implies \(\mu_t \subseteq \mu_s\). If \(x \in \mu_s\), then \(\mu(x) \geq s\), and so \(\mu(x) \geq t\) because \(\mu(x) \not< t\). Hence \(x \in \mu_t\), and \(\mu_s = \mu_t\). This completes the proof. \(\Box\)
Given a fuzzy $k$-ideal $\mu$ of $R$ we denote by $\text{Im}(\mu)$ the image set of $\mu$.

**Theorem 3.4.** Let $\mu$ be a fuzzy left (resp., right) $k$-ideal of $R$. If $\text{Im}(\mu) = \{t_1, t_2, \ldots, t_n\}$, where $t_1 < t_2 < \ldots < t_n$, then the family of left (resp., right) $k$-ideals $\mu_{t_i}$ $(i = 1, \ldots, n)$ constitutes the collection of all level left (resp., right) ideals of $\mu$.

**Proof.** If $t \in [0, 1]$ with $t < t_1$, then $\mu_{t_1} \subseteq \mu_t$. Since $\mu_{t_1} = R$ and $\mu_t = \mu_{t_1}$. If $t \in [0, 1]$ with $t_i < t < t_{i+1}$ $(1 \leq i \leq n - 1)$, then there is no $x \in R$ such that $t \leq \mu(x) < t_{i+1}$. It follows from Theorem 3.3 that $\mu_t = \mu_{t_{i+1}}$. This shows that for any $t \in L$ with $t \leq \mu(0)$, the level left (resp., right) ideal $\mu_t$ is in $\{\mu_{t_i} | 1 \leq i \leq n\}$. This completes the proof. \[ \square \]

Given any two sets $R$ and $S$, let $\mu$ be a fuzzy subset of $R$ and let $f : R \rightarrow S$ be any function. We define a fuzzy subset $\nu$ on $S$ by

$$
\nu(y) := \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, y \in S, \\
0 & \text{otherwise},
\end{cases}
$$

and we call $\nu$ the image of $\mu$ under $f$, written $f(\mu)$. For any fuzzy subset $\nu$ on $f(R)$, we define a fuzzy subset $\mu$ on $R$ by $\mu(x) := \nu(f(x))$ for all $x \in R$, and we call $\mu$ the preimage of $\nu$ under $f$ which is denoted by $f^{-1}(\nu)$.

**Theorem 3.5.** An onto homomorphic preimage of a fuzzy left (resp., right) $k$-ideal is a fuzzy left (resp., right) $k$-ideal.

**Proof.** Let $f : R \rightarrow S$ be an onto homomorphism. Let $\nu$ be a fuzzy left (resp., right) $k$-ideal on $S$ and let $\mu$ be the preimage of $\nu$ under $f$. Then it was proved that $\mu$ is a fuzzy left (resp., right) ideal of $R$ ([4]). For any $x, y \in S$, we have

$$
\mu(x) = \nu(f(x)) \\
\geq \min\{\max\{\nu(f(x) + f(y)), \nu(f(y) + f(x))\}, \nu(f(y))\} \\
= \min\{\max\{\nu(f(x + y)), \nu(f(y + x))\}, \nu(f(y))\} \\
= \min\{\max\{\mu(x + y), \mu(y + x)\}, \mu(y)\},
$$

proving that $\mu$ is a fuzzy left (resp., right) $k$-ideal of $R$. \[ \square \]
Proposition 3.6 ([4]). Let $f$ be a mapping from a set $X$ to a set $Y$, and let $\mu$ be a fuzzy subset of $X$. Then for every $t \in (0, 1]$,

$$\ (f(\mu))_t = \bigcap_{0 < s < t} f(\mu_{t-s}).$$

Theorem 3.7. Let $f : R \to S$ be an onto homomorphism and let $\mu$ be a fuzzy left (resp., right) $k$-ideal of $R$. Then the homomorphic image $f(\mu)$ of $\mu$ under $f$ is a fuzzy left (resp., right) $k$-ideal of $S$.

Proof. In view of Proposition 3.1 it is sufficient to show that each non-empty level subset of $f(\mu)$ is a left (resp., right) $k$-ideal of $S$. Let $(f(\mu))_t$ be a non-empty level subset of $f(\mu)$ for every $t \in [0, 1]$. If $t = 0$ then $(f(\mu))_t = S$. Assume $t \neq 0$. By Proposition 3.6, $(f(\mu))_t = \bigcap_{0 < s < t} f(\mu_{t-s})$. Hence $f(\mu_{t-s})$ is non-empty for each $0 < s < t$, and so $\mu_{t-s}$ is a non-empty level subset of $\mu$ for every $0 < s < t$. Since $\mu$ is a fuzzy left (resp., right) $k$-ideal of $R$, it follows from Proposition 3.1 that $\mu_{t-s}$ is a left (resp., right) $k$-ideal of $R$. Since $f$ is an onto homomorphism, $f(\mu_{t-s})$ is a left (resp., right) $k$-ideal of $S$. Hence $(f(\mu))_t$ being an intersection of a family of left (resp., right) $k$-ideals is also a left (resp., right) $k$-ideal of $S$. The proof is complete. \( \square \)

Definition 3.8. A left (resp., right) $k$-ideal $I$ of $R$ is said to be characteristic if $f(I) = I$ for all $f \in \text{Aut}(R)$, where $\text{Aut}(R)$ is the set of all automorphisms of $R$. A fuzzy left (resp., right) $k$-ideal $\mu$ of $R$ is said to be a fuzzy characteristic if $\mu(f(x)) = \mu(x)$ for all $x \in R$ and $f \in \text{Aut}(R)$.

Theorem 3.9. Let $\mu$ be a fuzzy left (resp., right) $k$-ideal of $R$ and let $f : R \to R$ be an onto homomorphism. Then the mapping $\mu^f : R \to [0, 1]$, defined by $\mu^f(x) := \mu(f(x))$ for all $x \in R$, is a fuzzy left (resp., right) $k$-ideal of $R$.

Proof. It was proved that $\mu^f$ is a fuzzy left (resp., right) $k$-ideal of $R$ ([4]). For any $x, y \in R$, we have

$$\mu^f(x) = \mu(f(x))$$

$$\geq \min\{\max\{\mu(f(x) + f(y)), \mu(f(y) + f(x))\}, \mu(f(y))\}$$

$$= \min\{\max\{\mu(f(x + y)), \mu(f(y + x))\}, \mu(f(y))\}$$

$$= \min\{\max\{\mu^f(x + y), \mu^f(y + x)\}, \mu^f(x)\},$$
proving that $\mu^f$ is a fuzzy left (resp., right) $k$-ideal of $R$.

**Theorem 3.10.** If $\mu$ is a fuzzy characteristic left (resp., right) $k$-ideal of $R$, then each level left (resp., right) $k$-ideal of $\mu$ is characteristic.

**Proof.** Let $\mu$ be a fuzzy characteristic left (resp., right) $k$-ideal of $R$ and let $f \in \text{Aut}(R)$. For any $t \in [0, 1]$, if $y \in f(\mu_t)$, then $\mu(y) = \mu(f(x)) = \mu(x) \geq t$ for some $x \in \mu_t$ with $y = f(x)$. It follows that $y \in \mu_t$. Conversely, if $y \in \mu_t$, then $t \leq \mu(y) = \mu(f(x)) = \mu(x)$ for some $x \in R$ with $y = f(x)$. It follows that $y \in f(\mu_t)$. This completes the proof. □

To prove the converse of Theorem 3.10, we need the following lemma.

**Lemma 3.11.** Let $\mu$ be a fuzzy left (resp., right) $k$-ideal of $R$ and let $x \in R$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \not\in \mu_s$ for all $s > t$.

**Proof.** Straightforward. □

**Theorem 3.12.** Let $\mu$ be a fuzzy left (resp., right) $k$-ideal of $R$. If each level left (resp. right) $k$-ideal of $\mu$ is characteristic, then $\mu$ is fuzzy characteristic.

**Proof.** Let $x \in R$ and $f \in \text{Aut}(R)$. If $\mu(x) = t \in [0, 1]$, then by Lemma 3.11 $x \in \mu_t$ and $x \not\in \mu_s$ for all $s > t$. Since each level left (resp., right) $k$-ideal of $\mu$ is characteristic, $f(x) \in f(\mu_t) = \mu_t$. Assume $\mu(f(x)) = s > t$. Then $f(x) \in \mu_s = f(\mu_s)$. Since $f$ is one-to-one, it follows that $x \in \mu_s$, a contradiction. Hence $\mu(f(x)) = t = \mu(x)$, showing that $\mu$ is fuzzy characteristic. □

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